Non-Parametric Trend Tests and Change-Point Detection

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1 Trend detection

1.1 Mann-Kendall Test

The non-parametric Mann-Kendall test is commonly employed to detect monotonic trends in series of environmental data, climate data or hydrological data. The null hypothesis, H_0, is that the data come from a population with independent realizations and are identically distributed. The alternative hypothesis, H_A, is that the data follow a monotonic trend. The Mann-Kendall test statistic is calculated according to :

$$S = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \operatorname{sgn}(X_j - X_k) \tag{1}$$

with

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases} \tag{2}$$

The mean of S is $E[S] = 0$ and the variance σ^2 is

$$\sigma^2 = \left\{ n(n-1)(2n+5) - \sum_{j=1}^{p} t_j (t_j - 1)(2t_j + 5) \right\} /18 \tag{3}$$

where p is the number of the tied groups in the data set and t_j is the number of data points in the jth tied group. The statistic S is approximately normal distributed provided that the following Z-transformation is employed:

$$Z = \begin{cases} \frac{S-1}{\sigma} & \text{if} \quad S > 0 \\ 0 & \text{if} \quad S = 0 \\ \frac{S+1}{\sigma} & \text{if} \quad S > 0 \end{cases} \tag{4}$$

The statistic S is closely related to Kendall's τ as given by:

$$\tau = \frac{S}{D} \tag{5}$$

where

$$D = \left[\frac{1}{2} n(n-1) - \frac{1}{2} \sum_{j=1}^{p} t_j (t_j - 1) \right]^{1/2} \left[\frac{1}{2} n(n-1) \right]^{1/2} \tag{6}$$

The univariate Mann-Kendall test is envoked as folllows:

```
> require(trend)
> data(maxau)
> Q <- maxau[,"Q"]
> mk.test(Q)
```

```
        Mann-Kendall trend test

data:  Q
z = -1.3989, n = 45, p-value = 0.1619
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
 -144.0000000 10450.0000000    -0.1454545
```

1.2 Seasonal Mann-Kendall Test

The Mann-Kendall statistic for the gth season is calculated as:

$$S_g = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \operatorname{sgn}\left(X_{jg} - X_{ig}\right), \quad g = 1, 2, \ldots, m \tag{7}$$

According to Hirsch *et al.* (1982), the seasonal Mann-Kendall statistic, $\hat{S}$, for the entire series is calculated according to

$$\hat{S} = \sum_{g=1}^{m} S_g \tag{8}$$

For further information, the reader is referred to Hipel and McLoed (1994, p. 866-869) and Hirsch *et al.* (1982). The seasonal Mann-Kendall test ist conducted as follows:

```
> require(trend)
> smk.test(nottem)

        Seasonal Mann-Kendall trend test (Hirsch-Slack test)

data:  nottem
z = 2.0919, p-value = 0.03645
alternative hypothesis: true S is not equal to 0
sample estimates:
    S  varS
  224 11364
```

Only the temperature data in Nottingham for August ($S = 80$, $p = 0.009$) as well as for September ($S = 67$, $p = 0.029$) show a significant ($p < 0.05$) positive trend according to the seasonal Mann-Kendall test. Thus, the global trend for the entire series is significant ($S = 224$, $p = 0.036$).

1.3 Correlated Seasonal Mann-Kendall Test

The correlated seasonal Mann-Kendall test can be employed, if the data are coreelated with e.g. the pre-ceeding months. For further information the reader is referred to Hipel and McLoed (1994, p. 869-871).

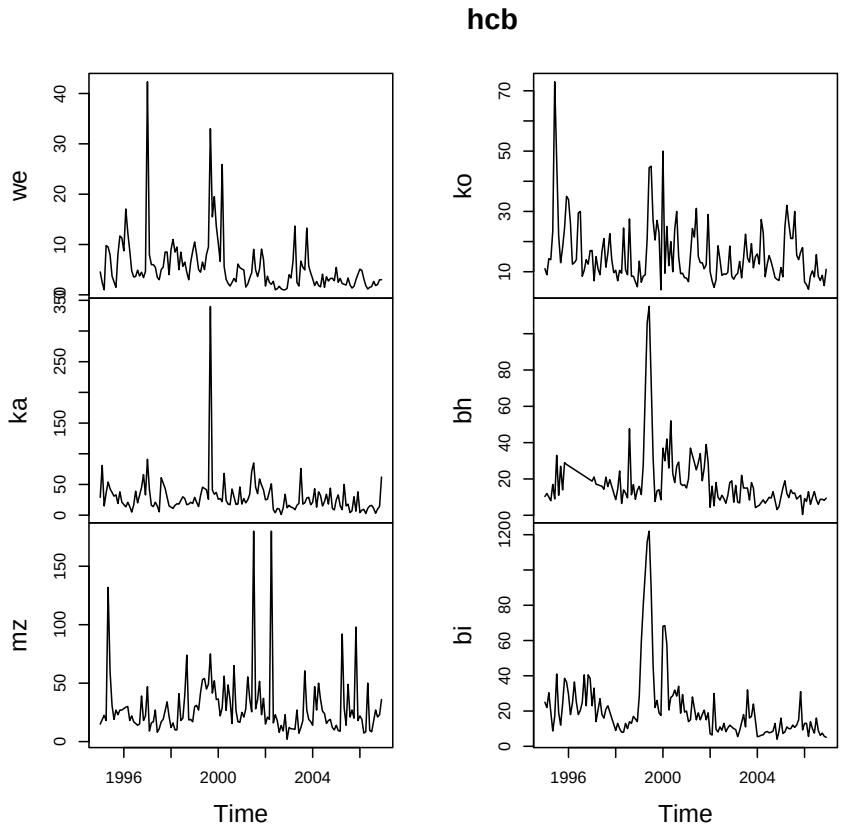
```
> require(trend)
> csmk.test(nottem)

        Correlated Seasonal Mann-Kendall Test

data:  nottem
z = 1.5974, p-value = 0.1102
alternative hypothesis: true S is not equal to 0
sample estimates:
       S     varS
  224.00 19663.33
```

1.4 Multivariate Mann-Kendall Test

Lettenmeier (1988) extended the Mann-Kendall test for trend to a multivariate or multi-site trend test. In this package the formulation of Libiseller and Grimvall (2002) is used for the test.

Particle bound Hexacholorobenzene (HCB, μg kg^{-}1) was monthly measured in suspended matter at six monitoring sites along the river strech of the River Rhine (Pohlert *et al.*, 2011). The below code-snippet tests for trend of each site and for the global trend at the multiple sites.

```
> require(trend)
> data(hcb)
> plot(hcb)
```




```
> ## Single site trends
> site <- c("we", "ka", "mz", "ko", "bh", "bi")
> for (i in 1:6) {print(site[i]) ; print(mk.test(hcb[,site[i]], continuity = TRUE))}

[1] "we"

        Mann-Kendall trend test

data:  hcb[, site[i]]
z = -5.8753, n = 144, p-value = 4.221e-09
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
-3.402000e+03  3.350867e+05 -3.317108e-01

[1] "ka"

        Mann-Kendall trend test
```

```
data:  hcb[, site[i]]
z = -3.5283, n = 144, p-value = 0.0004182
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
-2.043000e+03  3.349430e+05 -1.998191e-01

[1] "mz"

        Mann-Kendall trend test

data:  hcb[, site[i]]
z = -1.4447, n = 144, p-value = 0.1485
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
-8.370000e+02  3.348423e+05 -8.198541e-02

[1] "ko"

        Mann-Kendall trend test

data:  hcb[, site[i]]
z = -2.7916, n = 144, p-value = 0.005244
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
-1.617000e+03  3.350937e+05 -1.575802e-01

[1] "bh"

        Mann-Kendall trend test

data:  hcb[, site[i]]
z = -5.7681, n = 144, p-value = 8.018e-09
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
-3.340000e+03  3.350967e+05 -3.254744e-01

[1] "bi"

        Mann-Kendall trend test
```

```
data:  hcb[, site[i]]
z = -7.1165, n = 144, p-value = 1.107e-12
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
-4.120000e+03  3.350080e+05 -4.023498e-01
```

```
> ## Regional trend (all stations including covariance between stations
> mult.mk.test(hcb)

        Multivariate Mann-Kendall Trend Test

data:  hcb
z = -6.686, p-value = 2.293e-11
alternative hypothesis: true S is not equal to 0
sample estimates:
      S    varS
 -15359 5277014
```

1.5 Partial Mann-Kendall Test

This test can be conducted in the presence of co-variates. For full information, the reader is referred to Libiseller and Grimvall (2002).

We assume a correlation between concentration of suspended sediments (s) and flow at Maxau.

```
> data(maxau)
> s <- maxau[,"s"]; Q <- maxau[,"Q"]
> cor.test(s,Q, meth="spearman")

        Spearman's rank correlation rho

data:  s and Q
S = 10564, p-value = 0.0427
alternative hypothesis: true rho is not equal to 0
sample estimates:
      rho
0.3040843
```

As s is significantly positive related to flow, the partial Mann-Kendall test can be employed as follows.

```
> require(trend)
> data(maxau)
> s <- maxau[,"s"]; Q <- maxau[,"Q"]
> partial.mk.test(s,Q)
```

```
        Partial Mann-Kendall Trend Test

data:  t AND  s  .  Q
z = -3.597, p-value = 0.0003218
alternative hypothesis: true S is not equal to 0
sample estimates:
           S         varS          cor
-350.6576077 9503.2897820    0.3009888
```

The test indicates a highly significant decreasing trend ($S = -350.7$, $p < 0.001$) of s, when Q is partialled out.

1.6 Partial correlation trend test

This test performs a partial correlation trend test with either the Pearson's or the Spearman's correlation coefficients (r(tx.z)). The magnitude of the linear / monotonic trend with time is computed while the impact of the co-variate is partialled out.

```
> require(trend)
> data(maxau)
> s <- maxau[,"s"]; Q <- maxau[,"Q"]
> partial.cor.trend.test(s,Q, "spearman")

        Spearman's Partial Correlation Trend Test

data:  t AND  s  .  Q
t = -4.158, df = 43, p-value = 0.0001503
alternative hypothesis: true rho is not equal to 0
sample estimates:
   r(ts.Q)
-0.5355055
```

Likewise to the partial Mann-Kendall test, the partial correlation trend test using Spearman's correlation coefficient indicates a highly significant decreasing trend ($r_{S(ts.Q)} = -0.536$, $n = 45$, $p < 0.001$) of s when Q is partialled out.

1.7 Cox and Stuart Trend Test

The non-parametric Cox and Stuart Trend test tests the first third of the series with the last third for trend.

```
> ## Example from Schoenwiese (1992, p. 114)
> ## Number of frost days in April at Munich from 1957 to 1968
> ## z = -0.5, Accept H0
> frost <- ts(data=c(9,12,4,3,0,4,2,1,4,2,9,7), start=1957)
> cs.test(frost)
```

```
        Cox and Stuart Trend test

data:  frost
z = -0.5, n = 12, p-value = 0.6171
alternative hypothesis: monotonic trend

> ## Example from Sachs (1997, p. 486-487)
> ## z ~ 2.1, Reject H0 on a level of p = 0.0357
> x <- c(5,6,2,3,5,6,4,3,7,8,9,7,5,3,4,7,3,5,6,7,8,9)
> cs.test(x)

        Cox and Stuart Trend test

data:  x
z = 2.0926, n = 22, p-value = 0.03639
alternative hypothesis: monotonic trend
```

2 Magnitude of trend

2.1 Sen's slope

This test computes both the slope (i.e. linear rate of change) and intercept according to Sen's method. First, a set of linear slopes is calculated as follows:

$$d_k = \frac{X_j - X_i}{j - i} \tag{9}$$

for $(1 \leq i < j \leq n)$, where d is the slope, X denotes the variable, n is the number of data, and i, j are indices.

Sen's slope is then calculated as the median from all slopes: $b = \text{Median}\, d_k$. The intercepts are computed for each timestep t as given by

$$a_t = X_t - b * t \tag{10}$$

and the corresponding intercept is as well the median of all intercepts.

This function also computes the upper and lower confidence limits for sens slope.

```
> require(trend)
> s <- maxau[,"s"]
> sens.slope(s)

        Sen's slope

data:  s
z = -3.8445, n = 45, p-value = 0.0001208
alternative hypothesis: true z is not equal to 0
```

```
95 percent confidence interval:
 -0.4196477 -0.1519026
sample estimates:
Sen's slope
 -0.2876139
```

2.2 Seasonal Sen's slope

Acccording to Hirsch *et al.* (1982) the seasonal Sen's slope is calculated as follows:

$$d_{ijk} = \frac{X_{ij} - x_{ik}}{j - k} \tag{11}$$

for each (x_{ij}, x_{ik} pair $i = 1, 2, \ldots, m$, where $1 \leq k < j \leq n_i$ and n_i is the number of known values in the ith season. The seasonal slope estimator is the median of the d_{ijk} values.

```
> require(trend)
> sea.sens.slope(nottem)

[1] 0.05
```

3 Change-point detection

3.1 Pettitt's test

The approach after Pettitt (1979) is commonly applied to detect a single change-point in hydrological series or climate series with continuous data. It tests the H_0: The T variables follow one or more distributions that have the same location parameter (no change), against the alternative: a change point exists. The non-parametric statistic is defined as:

$$K_T = \max |U_{t,T}| , \tag{12}$$

where

$$U_{t,T} = \sum_{i=1}^{t} \sum_{j=t+1}^{T} \operatorname{sgn} (X_i - X_j) \tag{13}$$

The change-point of the series is located at K_T, provided that the statistic is significant. The significance probability of K_T is approximated for $p \leq 0.05$ with

$$p \simeq 2 \ \exp \left(\frac{-6 \ K_T^2}{T^3 + T^2} \right) \tag{14}$$

The Pettitt-test is conducted in such a way:

```
> require(trend)
> data(PagesData)
> pettitt.test(PagesData)

        Pettitt's test for single change-point detection

data:  PagesData
U* = 232, p-value = 0.01456
alternative hypothesis: two.sided
sample estimates:
probable change point at time K
                             17
```

As given in the publication of Pettitt (1979) the change-point of Page's data is located at $t = 17$, with $K_T = 232$ and $p = 0.014$.

3.2 Buishand Range Test

Let X denote a normal random variate, then the following model with a single shift (change-point) can be proposed:

$$x_i = \begin{cases} \mu + \epsilon_i, & i = 1, \ldots, m \\ \mu + \Delta + \epsilon_i & i = m+1, \ldots, n \end{cases} \tag{15}$$

$\epsilon \approx N(0, \sigma)$. The null hypothesis $\Delta = 0$ is tested against the alternative $\Delta \neq 0$.

In the Buishand range test (Buishand, 1982), the rescaled adjusted partial sums are calculated as

$$S_k = \sum_{i=1}^{k} (x_i - \hat{x}) \qquad (1 \leq i \leq n) \tag{16}$$

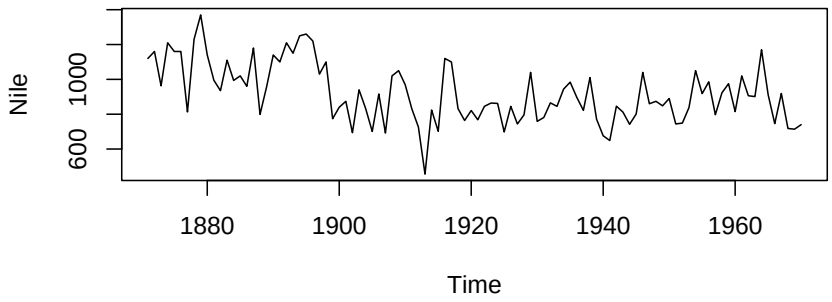
The test statistic is calculated as:

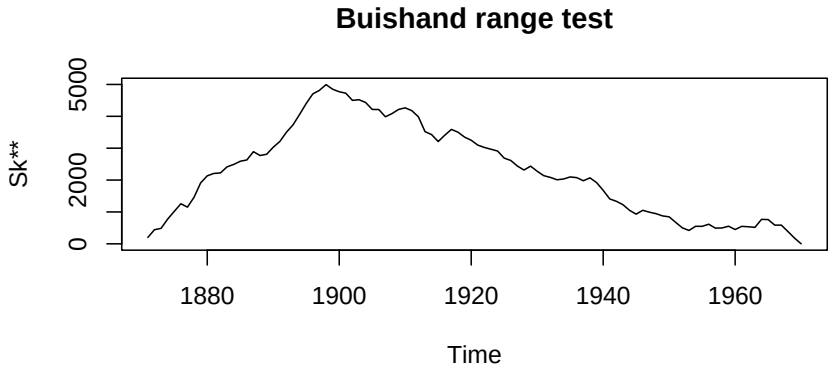
$$Rb = \frac{\max S_k - \min S_k}{\sigma} \tag{17}$$

the p.value is estimated with a Monte Carlo simulation using m replicates.

```
> require(trend)
> (res <- br.test(Nile))

        Buishand range test

data:  Nile
R / sqrt(n) = 2.9518, n = 100, p-value < 2.2e-16
alternative hypothesis: true delta is not equal to 0
sample estimates:
probable change point at time K
                             28
```

```
> par(mfrow=c(2,1))
> plot(Nile); plot(res)
```





3.3 Buishand U Test

In the Buishand U Test (Buishand, 1984), the null hypothesis is the same as in the Buishand Range Test (see Eq. 15). The test statistic is

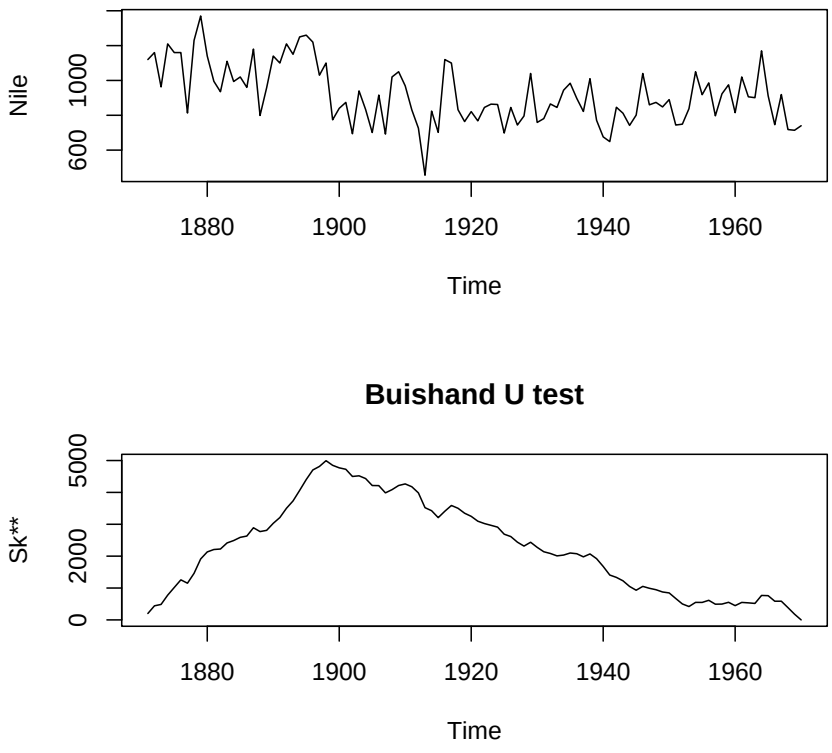
$$U = [n\,(n+1)]^{-1} \sum_{k=1}^{n-1} (S_k/D_x)^2 \tag{18}$$

with

$$D_x = \sqrt{n^{-1} \sum_{i=1}^{n} (x_i - \bar{x})} \tag{19}$$

and S_k as given in Eq. 16. The p.value is estimated with a Monte Carlo simulation using m replicates.

```
> require(trend)
> (res <- bu.test(Nile))

        Buishand U test

data:  Nile
U = 2.4764, n = 100, p-value < 2.2e-16
alternative hypothesis: true delta is not equal to 0
sample estimates:
probable change point at time K
                             28

> par(mfrow=c(2,1))
> plot(Nile); plot(res)
```



3.4 Standard Normal Homogeinity Test

In the Standard Normal Homogeinity Test (Alexandersson, 1986), the null hypothesis is the same as in the Buishand Range Test (see Eq. 15). The test statistic is

$$T_k = kz_1^2 + (n-k)\, z_2^2 \qquad (1 \leq k < n) \tag{20}$$

where

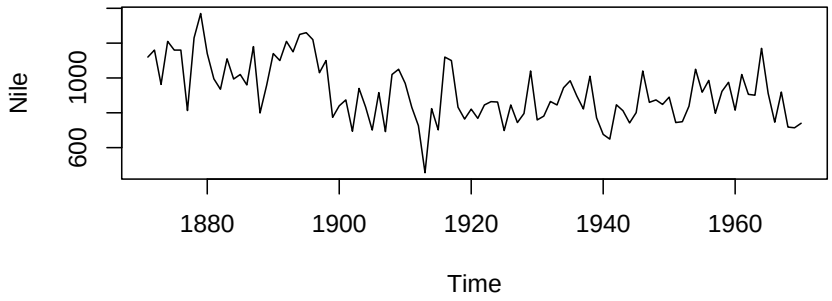
$$z_1 = \frac{1}{k} \sum_{i=1}^{k} \frac{x_i - \bar{x}}{\sigma} \quad z_2 = \frac{1}{n-k} \sum_{i=k+1}^{n} \frac{x_i - \bar{x}}{\sigma}. \tag{21}$$

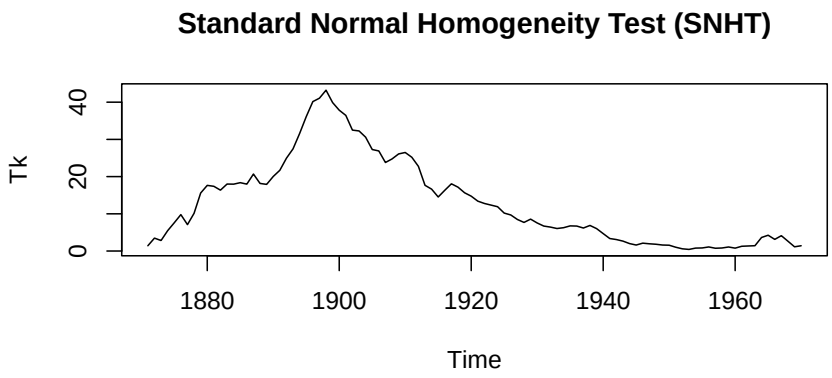
The critical value is:

$$T = \max T_k \tag{22}$$

The p.value is estimated with a Monte Carlo simulation using m replicates.

```
> require(trend)
> (res <- snh.test(Nile))

        Standard Normal Homogeneity Test (SNHT)

data:  Nile
T = 43.219, n = 100, p-value < 2.2e-16
alternative hypothesis: true delta is not equal to 0
sample estimates:
probable change point at time K
                             28

> par(mfrow=c(2,1))
> plot(Nile); plot(res)
```





4 Randomness

4.1 Wallis and Moore phase-frequency test

A phase frequency test was proposed by Wallis and Moore (1941) and is used for testing a series for randomness:

```
> ## Example from Schoenwiese (1992, p. 113)
> ## Number of frost days in April at Munich from 1957 to 1968
> ## z = -0.124, Accept H0
> frost <- ts(data=c(9,12,4,3,0,4,2,1,4,2,9,7), start=1957)
> wm.test(frost)

        Wallis and Moore Phase-Frequency test

data:  frost
z = -0.12384, p-value = 0.9014
alternative hypothesis: The series is significantly different from randomness
```

```
> ## Example from Sachs (1997, p. 486)
> ## z = 2.56, Reject H0 on a level of p < 0.05
> x <- c(5,6,2,3,5,6,4,3,7,8,9,7,5,3,4,7,3,5,6,7,8,9)
> wm.test(x)

        Wallis and Moore Phase-Frequency test

data:  x
z = 2.5513, p-value = 0.01073
alternative hypothesis: The series is significantly different from randomness
```

4.2 Bartels test for randomness

Bartels (1982) has proposed a rank version of von Neumann's ratio test for testing a series for randomness:

```
> ## Example from Schoenwiese (1992, p. 113)
> ## Number of frost days in April at Munich from 1957 to 1968
> ##
> frost <- ts(data=c(9,12,4,3,0,4,2,1,4,2,9,7), start=1957)
> bartels.test(frost)

        Bartels's test for randomness

data:  frost
RVN = 1.3304, p-value = 0.1137
alternative hypothesis: The series is significantly different from randomness

> ## Example from Sachs (1997, p. 486)
> x <- c(5,6,2,3,5,6,4,3,7,8,9,7,5,3,4,7,3,5,6,7,8,9)
> bartels.test(x)

        Bartels's test for randomness

data:  x
RVN = 1.0444, p-value = 0.008371
alternative hypothesis: The series is significantly different from randomness

> ## Example from Bartels (1982, p. 43)
> x <- c(4, 7, 16, 14, 12, 3, 9, 13, 15, 10, 6, 5, 8, 2, 1, 11, 18, 17)
> bartels.test(x)

        Bartels's test for randomness

data:  x
RVN = 0.97626, p-value = 0.009463
alternative hypothesis: The series is significantly different from randomness
```

4.3 Wald-Wolfowitz test for stationarity and independence

Wald and Wolfowitz (1942) have proposed a test for randomness:

```
> ## Example from Schoenwiese (1992, p. 113)
> ## Number of frost days in April at Munich from 1957 to 1968
> ##
> frost <- ts(data=c(9,12,4,3,0,4,2,1,4,2,9,7), start=1957)
> ww.test(frost)

        Wald-Wolfowitz test for independence and stationarity

data:  frost
z = 1.9198, n = 12, p-value = 0.05488
alternative hypothesis: The series is significantly different from
 independence and stationarity

> ## Example from Sachs (1997, p. 486)
> x <- c(5,6,2,3,5,6,4,3,7,8,9,7,5,3,4,7,3,5,6,7,8,9)
> ww.test(x)

        Wald-Wolfowitz test for independence and stationarity

data:  x
z = 2.1394, n = 22, p-value = 0.03241
alternative hypothesis: The series is significantly different from
 independence and stationarity

> ## Example from Bartels (1982, p. 43)
> x <- c(4, 7, 16, 14, 12, 3, 9, 13, 15, 10, 6, 5, 8, 2, 1, 11, 18, 17)
> ww.test(x)

        Wald-Wolfowitz test for independence and stationarity

data:  x
z = 1.7304, n = 18, p-value = 0.08357
alternative hypothesis: The series is significantly different from
 independence and stationarity
```

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